

# HYPOTHESIS TESTING OF SINGLE MEAN AND SINGLE PROPORTION: EXAMPLES\*

Susan Dean  
Barbara Illowsky, Ph.D.

This work is produced by The Connexions Project and licensed under the Creative Commons Attribution License <sup>†</sup>

## Abstract

This module provides examples of Hypothesis Testing of a Single Mean and a Single Proportion as a part of the Collaborative Statistics collection (coll0522) by Barbara Illowsky and Susan Dean.

### Example 1

Jeffrey, as an eight-year old, **established a mean time of 16.43 seconds** for swimming the 25-yard freestyle, with a **standard deviation of 0.8 seconds**. His dad, Frank, thought that Jeffrey could swim the 25-yard freestyle faster by using goggles. Frank bought Jeffrey a new pair of expensive goggles and timed Jeffrey for **15 25-yard freestyle swims**. For the 15 swims, **Jeffrey's mean time was 16 seconds. Frank thought that the goggles helped Jeffrey to swim faster than the 16.43 seconds**. Conduct a hypothesis test using a preset  $\alpha = 0.05$ . Assume that the swim times for the 25-yard freestyle are normal.

### Solution

Set up the Hypothesis Test:

Since the problem is about a mean, this is a **test of a single population mean**.

$H_o: \mu = 16.43$        $H_a: \mu < 16.43$

For Jeffrey to swim faster, his time will be less than 16.43 seconds. The "<" tells you this is left-tailed.

Determine the distribution needed:

**Random variable:**  $\bar{X}$  = the mean time to swim the 25-yard freestyle.

**Distribution for the test:**  $\bar{X}$  is normal (population **standard deviation** is known:  $\sigma = 0.8$ )

$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$       Therefore,  $\bar{X} \sim N\left(16.43, \frac{0.8}{\sqrt{15}}\right)$

$\mu = 16.43$  comes from  $H_o$  and not the data.  $\sigma = 0.8$ , and  $n = 15$ .

Calculate the p-value using the normal distribution for a mean:

p-value =  $P(\bar{x} < 16) = 0.0187$  where the sample mean in the problem is given as 16.

p-value = 0.0187 (This is called the **actual level of significance**.) The p-value is the area to the left of the sample mean is given as 16.

**Graph:**

---

\*Version 1.25: Jun 18, 2012 9:40 am -0500

<sup>†</sup><http://creativecommons.org/licenses/by/3.0/>

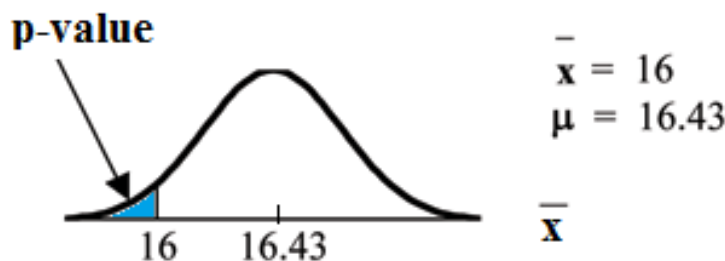


Figure 1

$\mu = 16.43$  comes from  $H_o$ . Our assumption is  $\mu = 16.43$ .

**Interpretation of the p-value:** If  $H_o$  is true, there is a 0.0187 probability (1.87%) that Jeffrey's mean time to swim the 25-yard freestyle is 16 seconds or less. Because a 1.87% chance is small, the mean time of 16 seconds or less is unlikely to have happened randomly. It is a rare event.

Compare  $\alpha$  and the p-value:

$$\alpha = 0.05 \quad \text{p-value} = 0.0187 \quad \alpha > \text{p-value}$$

**Make a decision:** Since  $\alpha > \text{p-value}$ , reject  $H_o$ .

This means that you reject  $\mu = 16.43$ . In other words, you do not think Jeffrey swims the 25-yard freestyle in 16.43 seconds but faster with the new goggles.

**Conclusion:** At the 5% significance level, we conclude that Jeffrey swims faster using the new goggles. The sample data show there is sufficient evidence that Jeffrey's mean time to swim the 25-yard freestyle is less than 16.43 seconds.

The p-value can easily be calculated using the TI-83+ and the TI-84 calculators:

Press STAT and arrow over to TESTS. Press 1:Z-Test. Arrow over to Stats and press ENTER. Arrow down and enter 16.43 for  $\mu_0$  (null hypothesis), .8 for  $\sigma$ , 16 for the sample mean, and 15 for  $n$ . Arrow down to  $\mu$ : (alternate hypothesis) and arrow over to  $<\mu_0$ . Press ENTER. Arrow down to Calculate and press ENTER. The calculator not only calculates the p-value ( $p = 0.0187$ ) but it also calculates the test statistic (z-score) for the sample mean.  $\mu < 16.43$  is the alternate hypothesis. Do this set of instructions again except arrow to Draw (instead of Calculate). Press ENTER. A shaded graph appears with  $z = -2.08$  (test statistic) and  $p = 0.0187$  (p-value). Make sure when you use Draw that no other equations are highlighted in  $Y =$  and the plots are turned off.

When the calculator does a Z-Test, the Z-Test function finds the p-value by doing a normal probability calculation using the **Central Limit Theorem**:

$$P(\bar{x} < 16) = 2\text{nd DISTR normcdf}(-10^{99}, 16, 16.43, 0.8/\sqrt{15}).$$

The Type I and Type II errors for this problem are as follows:

The Type I error is to conclude that Jeffrey swims the 25-yard freestyle, on average, in less than 16.43 seconds when, in fact, he actually swims the 25-yard freestyle, on average, in 16.43 seconds. (Reject the null hypothesis when the null hypothesis is true.)

The Type II error is that there is not evidence to conclude that Jeffrey swims the 25-yard free-style, on average, in less than 16.43 seconds when, in fact, he actually does swim the 25-yard free-style, on average, in less than 16.43 seconds. (Do not reject the null hypothesis when the null hypothesis is false.)

**Historical Note:** The traditional way to compare the two probabilities,  $\alpha$  and the p-value, is to compare the critical value (z-score from  $\alpha$ ) to the test statistic (z-score from data). The calculated test statistic for the p-value is  $-2.08$ . (From the Central Limit Theorem, the test statistic formula is  $z = \frac{\bar{x} - \mu_X}{\left(\frac{\sigma_X}{\sqrt{n}}\right)}$ . For this problem,  $\bar{x} = 16$ ,  $\mu_X = 16.43$  from the null hypothesis,  $\sigma_X = 0.8$ , and  $n = 15$ .) You can find the critical value for  $\alpha = 0.05$  in the normal table (see **15.Tables** in the Table of Contents). The z-score for an area to the left equal to 0.05 is midway between  $-1.65$  and  $-1.64$  ( $0.05$  is midway between  $0.0505$  and  $0.0495$ ). The z-score is  $-1.645$ . Since  $-1.645 > -2.08$  (which demonstrates that  $\alpha > \text{p-value}$ ), reject  $H_o$ . Traditionally, the decision to reject or not reject was done in this way. Today, comparing the two probabilities  $\alpha$  and the p-value is very common. For this problem, the p-value,  $0.0187$  is considerably smaller than  $\alpha$ ,  $0.05$ . You can be confident about your decision to reject. The graph shows  $\alpha$ , the p-value, and the test statistics and the critical value.

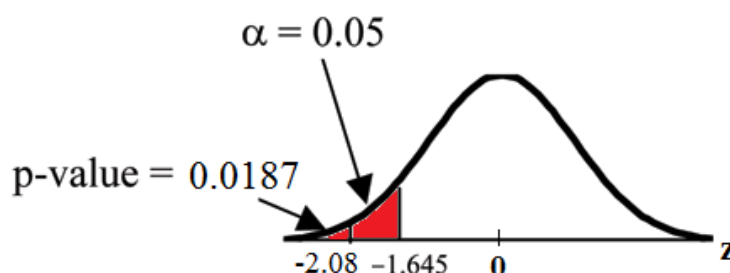


Figure 2

### Example 2

A college football coach thought that his players could bench press a **mean weight of 275 pounds**. It is known that the **standard deviation is 55 pounds**. Three of his players thought that the mean weight was **more than** that amount. They asked **30** of their teammates for their estimated maximum lift on the bench press exercise. The data ranged from 205 pounds to 385 pounds. The actual different weights were (frequencies are in parentheses) 205(3); 215(3); 225(1); 241(2); 252(2); 265(2); 275(2); 313(2); 316(5); 338(2); 341(1); 345(2); 368(2); 385(1). (Source: data from Reuben Davis, Kraig Evans, and Scott Gunderson.)

Conduct a hypothesis test using a 2.5% level of significance to determine if the bench press mean is **more than 275 pounds**.

### Solution

Set up the Hypothesis Test:

Since the problem is about a mean weight, this is a **test of a single population mean**.

$H_o: \mu = 275$      $H_a: \mu > 275$     This is a right-tailed test.

Calculating the distribution needed:

Random variable:  $\bar{X}$  = the mean weight, in pounds, lifted by the football players.

**Distribution for the test:** It is normal because  $\sigma$  is known.

$\bar{X} \sim N\left(275, \frac{55}{\sqrt{30}}\right)$

$\bar{x} = 286.2$  pounds (from the data).

$\sigma = 55$  pounds (**Always use  $\sigma$  if you know it.**) We assume  $\mu = 275$  pounds unless our data shows us otherwise.

Calculate the p-value using the normal distribution for a mean and using the sample mean as input (see the calculator instructions below for using the data as input):

$$p\text{-value} = P(\bar{x} > 286.2) = 0.1323.$$

**Interpretation of the p-value:** If  $H_o$  is true, then there is a 0.1331 probability (13.23%) that the football players can lift a mean weight of 286.2 pounds or more. Because a 13.23% chance is large enough, a mean weight lift of 286.2 pounds or more is not a rare event.

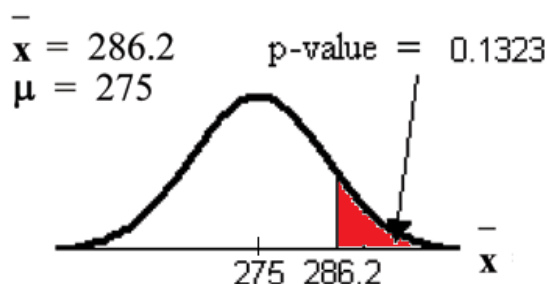


Figure 3

Compare  $\alpha$  and the p-value:

$$\alpha = 0.025 \quad p\text{-value} = 0.1323$$

**Make a decision:** Since  $\alpha < p\text{-value}$ , do not reject  $H_o$ .

**Conclusion:** At the 2.5% level of significance, from the sample data, there is not sufficient evidence to conclude that the true mean weight lifted is more than 275 pounds.

The p-value can easily be calculated using the TI-83+ and the TI-84 calculators:

Put the data and frequencies into lists. Press **STAT** and arrow over to **TESTS**. Press **1:Z-Test**. Arrow over to **Data** and press **ENTER**. Arrow down and enter 275 for  $\mu_0$ , 55 for  $\sigma$ , the name of the list where you put the data, and the name of the list where you put the frequencies. Arrow down to  $\mu :$  and arrow over to  $> \mu_0$ . Press **ENTER**. Arrow down to **Calculate** and press **ENTER**. The calculator not only calculates the p-value ( $p = 0.1331$ , a little different from the above calculation - in it we used the sample mean rounded to one decimal place instead of the data) but it also calculates the test statistic (z-score) for the sample mean, the sample mean, and the sample standard deviation.  $\mu > 275$  is the alternate hypothesis. Do this set of instructions again except arrow to **Draw** (instead of **Calculate**). Press **ENTER**. A shaded graph appears with  $z = 1.112$  (test statistic) and  $p = 0.1331$  (p-value). Make sure when you use **Draw** that no other equations are highlighted in  $Y =$  and the plots are turned off.

### Example 3

Statistics students believe that the mean score on the first statistics test is 65. A statistics instructor thinks the mean score is higher than 65. He samples ten statistics students and obtains the scores 65; 65; 70; 67; 66; 63; 63; 68; 72; 71. He performs a hypothesis test using a 5% level of significance. The data are from a normal distribution.

### Solution

Set up the Hypothesis Test:

A 5% level of significance means that  $\alpha = 0.05$ . This is a test of a **single population mean**.

$$H_o: \mu = 65 \quad H_a: \mu > 65$$

Since the instructor thinks the average score is higher, use a "> ". The "> " means the test is right-tailed.

Determine the distribution needed:

**Random variable:**  $\bar{X}$  = average score on the first statistics test.

**Distribution for the test:** If you read the problem carefully, you will notice that there is **no population standard deviation given**. You are only given  $n = 10$  sample data values. Notice also that the data come from a normal distribution. This means that the distribution for the test is a student's-t.

Use  $t_{df}$ . Therefore, the distribution for the test is  $t_9$  where  $n = 10$  and  $df = 10 - 1 = 9$ .

Calculate the p-value using the Student's-t distribution:

p-value =  $P(\bar{x} > 67) = 0.0396$  where the sample mean and sample standard deviation are calculated as 67 and 3.1972 from the data.

**Interpretation of the p-value:** If the null hypothesis is true, then there is a 0.0396 probability (3.96%) that the sample mean is 67 or more.

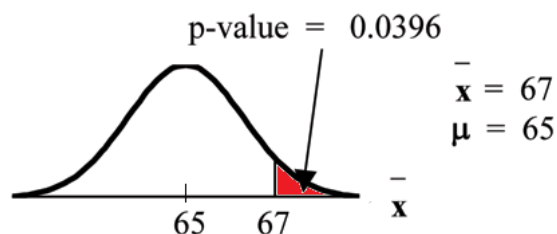


Figure 4

Compare  $\alpha$  and the p-value:

Since  $\alpha = .05$  and p-value = 0.0396. Therefore,  $\alpha > \text{p-value}$ .

**Make a decision:** Since  $\alpha > \text{p-value}$ , reject  $H_o$ .

This means you reject  $\mu = 65$ . In other words, you believe the average test score is more than 65.

**Conclusion:** At a 5% level of significance, the sample data show sufficient evidence that the mean (average) test score is more than 65, just as the math instructor thinks.

The p-value can easily be calculated using the TI-83+ and the TI-84 calculators:

Put the data into a list. Press **STAT** and arrow over to **TESTS**. Press **2:T-Test**. Arrow over to **Data** and press **ENTER**. Arrow down and enter 65 for  $\mu_0$ , the name of the list where you put the data, and 1 for **Freq**:. Arrow down to  $\mu$  : and arrow over to  $> \mu_0$ . Press **ENTER**. Arrow down to **Calculate** and press **ENTER**. The calculator not only calculates the p-value ( $p = 0.0396$ ) but it also calculates the test statistic (t-score) for the sample mean, the sample mean, and the sample standard deviation.  $\mu > 65$  is the alternate hypothesis. Do this set of instructions again except arrow to **Draw** (instead of **Calculate**). Press **ENTER**. A shaded graph appears with  $t = 1.9781$  (test statistic) and  $p = 0.0396$  (p-value). Make sure when you use **Draw** that no other equations are highlighted in  $Y =$  and the plots are turned off.

**Example 4**

Joon believes that 50% of first-time brides in the United States are younger than their grooms. She performs a hypothesis test to determine if the percentage is **the same or different from 50%**. Joon samples **100 first-time brides** and **53** reply that they are younger than their grooms. For the hypothesis test, she uses a 1% level of significance.

**Solution**

Set up the Hypothesis Test:

The 1% level of significance means that  $\alpha = 0.01$ . This is a **test of a single population proportion**.

$$H_o: p = 0.50 \quad H_a: p \neq 0.50$$

The words "**is the same or different from**" tell you this is a two-tailed test.

Calculate the distribution needed:

**Random variable:**  $P'$  = the percent of of first-time brides who are younger than their grooms.

**Distribution for the test:** The problem contains no mention of a mean. The information is given in terms of percentages. Use the distribution for  $P'$ , the estimated proportion.

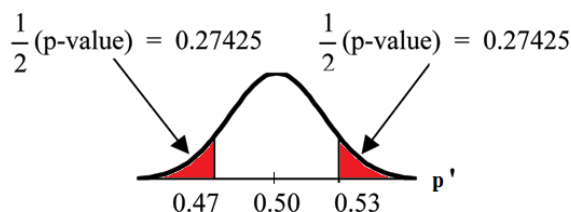
$P' \sim N\left(p, \sqrt{\frac{p \cdot q}{n}}\right)$  Therefore,  $P' \sim N\left(0.5, \sqrt{\frac{0.5 \cdot 0.5}{100}}\right)$  where  $p = 0.50$ ,  $q = 1 - p = 0.50$ , and  $n = 100$ .

Calculate the p-value using the normal distribution for proportions:

$$\text{p-value} = P(p' < 0.47 \text{ or } p' > 0.53) = 0.5485$$

$$\text{where } x = 53, p' = \frac{x}{n} = \frac{53}{100} = 0.53.$$

**Interpretation of the p-value:** If the null hypothesis is true, there is 0.5485 probability (54.85%) that the sample (estimated) proportion  $p'$  is 0.53 or more OR 0.47 or less (see the graph below).



**Figure 5**

$\mu = p = 0.50$  comes from  $H_o$ , the null hypothesis.

$p' = 0.53$ . Since the curve is symmetrical and the test is two-tailed, the  $p'$  for the left tail is equal to  $0.50 - 0.03 = 0.47$  where  $\mu = p = 0.50$ . (0.03 is the difference between 0.53 and 0.50.)

Compare  $\alpha$  and the p-value:

Since  $\alpha = 0.01$  and  $\text{p-value} = 0.5485$ . Therefore,  $\alpha < \text{p-value}$ .

**Make a decision:** Since  $\alpha < \text{p-value}$ , you cannot reject  $H_o$ .

**Conclusion:** At the 1% level of significance, the sample data do not show sufficient evidence that the percentage of first-time brides that are younger than their grooms is different from 50%.

The p-value can easily be calculated using the TI-83+ and the TI-84 calculators:

Press STAT and arrow over to TESTS. Press 5:1-PropZTest. Enter .5 for  $p_0$ , 53 for  $x$  and 100 for  $n$ . Arrow down to Prop and arrow to **not equals**  $p_0$ . Press ENTER. Arrow down to Calculate and press ENTER. The calculator calculates the p-value ( $p = 0.5485$ ) and the test statistic (z-score). Prop **not equals** .5 is the alternate hypothesis. Do this set of instructions again except arrow to

**Draw** (instead of **Calculate**). Press **ENTER**. A shaded graph appears with  $z = 0.6$  (test statistic) and  $p = 0.5485$  (p-value). Make sure when you use **Draw** that no other equations are highlighted in  $Y =$  and the plots are turned off.

The Type I and Type II errors are as follows:

The Type I error is to conclude that the proportion of first-time brides that are younger than their grooms is different from 50% when, in fact, the proportion is actually 50%. (Reject the null hypothesis when the null hypothesis is true).

The Type II error is there is not enough evidence to conclude that the proportion of first time brides that are younger than their grooms differs from 50% when, in fact, the proportion does differ from 50%. (Do not reject the null hypothesis when the null hypothesis is false.)

## Example 5

### Problem 1

Suppose a consumer group suspects that the proportion of households that have three cell phones is 30%. A cell phone company has reason to believe that the proportion is 30%. Before they start a big advertising campaign, they conduct a hypothesis test. Their marketing people survey 150 households with the result that 43 of the households have three cell phones.

### Solution

Set up the Hypothesis Test:

$$H_o: p = 0.30 \quad H_a: p \neq 0.30$$

Determine the distribution needed:

The **random variable** is  $P' =$  proportion of households that have three cell phones.

The **distribution** for the hypothesis test is  $P' \sim N\left(0.30, \sqrt{\frac{(0.30) \cdot (0.70)}{150}}\right)$

### Problem 2

The value that helps determine the p-value is  $p'$ . Calculate  $p'$ .

### Problem 3

What is a **success** for this problem?

### Problem 4

What is the level of significance?

Draw the graph for this problem. Draw the horizontal axis. Label and shade appropriately.

### Problem 5

Calculate the p-value.

### Problem 6

Make a decision. \_\_\_\_\_ (Reject/Do not reject)  $H_0$  because \_\_\_\_\_.

The next example is a poem written by a statistics student named Nicole Hart. The solution to the problem follows the poem. Notice that the hypothesis test is for a single population proportion. This means that the null and alternate hypotheses use the parameter  $p$ . The distribution for the test is normal. The estimated proportion  $p'$  is the proportion of fleas killed to the total fleas found on Fido. This is sample information. The problem gives a preconceived  $\alpha = 0.01$ , for comparison, and a 95% confidence interval computation. The poem is clever and humorous, so please enjoy it!

NOTE: Hypothesis testing problems consist of multiple steps. To help you do the problems, solution sheets are provided for your use. Look in the Table of Contents Appendix for the topic "Solution Sheets." If you like, use copies of the appropriate solution sheet for homework problems.

### Example 6

My dog has so many fleas,  
They do not come off with ease.  
As for shampoo, I have tried many types  
Even one called Bubble Hype,  
Which only killed 25% of the fleas,  
Unfortunately I was not pleased.

I've used all kinds of soap,  
Until I had give up hope  
Until one day I saw  
An ad that put me in awe.

A shampoo used for dogs  
Called GOOD ENOUGH to Clean a Hog  
Guaranteed to kill more fleas.

I gave Fido a bath  
And after doing the math  
His number of fleas  
Started dropping by 3's!

Before his shampoo  
I counted 42.  
At the end of his bath,  
I redid the math  
And the new shampoo had killed 17 fleas.  
So now I was pleased.

Now it is time for you to have some fun  
With the level of significance being .01,  
You must help me figure out  
Use the new shampoo or go without?

### Solution

Set up the Hypothesis Test:

$$H_o: p = 0.25 \quad H_a: p > 0.25$$

Determine the distribution needed:

In words, CLEARLY state what your random variable  $\bar{X}$  or  $P'$  represents.

$P'$  = The proportion of fleas that are killed by the new shampoo

State the distribution to use for the test.

$$\text{Normal: } N\left(0.25, \sqrt{\frac{(0.25)(1-0.25)}{42}}\right)$$

$$\text{Test Statistic: } z = 2.3163$$

Calculate the p-value using the normal distribution for proportions:

$$\text{p-value} = 0.0103$$



In 1 – 2 complete sentences, explain what the p-value means for this problem.

If the null hypothesis is true (the proportion is 0.25), then there is a 0.0103 probability that the sample (estimated) proportion is 0.4048 ( $\frac{17}{42}$ ) or more.

Use the previous information to sketch a picture of this situation. CLEARLY, label and scale the horizontal axis and shade the region(s) corresponding to the p-value.

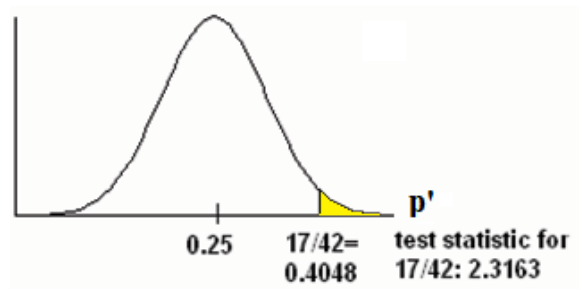


Figure 6

Compare  $\alpha$  and the p-value:

Indicate the correct decision (“reject” or “do not reject” the null hypothesis), the reason for it, and write an appropriate conclusion, using COMPLETE SENTENCES.

alpha	decision	reason for decision
0.01	Do not reject $H_o$	$\alpha < \text{p-value}$

Table 1

**Conclusion:** At the 1% level of significance, the sample data do not show sufficient evidence that the percentage of fleas that are killed by the new shampoo is more than 25%.

Construct a 95% Confidence Interval for the true mean or proportion. Include a sketch of the graph of the situation. Label the point estimate and the lower and upper bounds of the Confidence Interval.

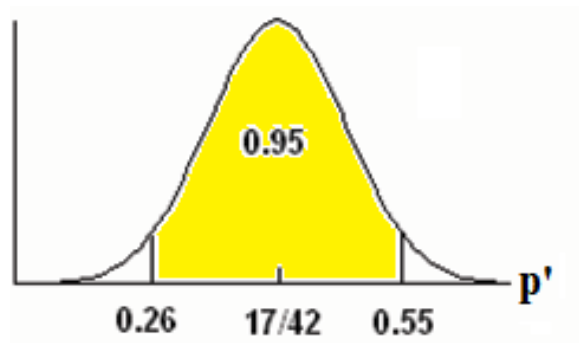


Figure 7

**Confidence Interval:** (0.26, 0.55) We are 95% confident that the true population proportion  $p$  of fleas that are killed by the new shampoo is between 26% and 55%.

NOTE: This test result is not very definitive since the p-value is very close to alpha. In reality, one would probably do more tests by giving the dog another bath after the fleas have had a chance to return.

## Glossary

### Definition 1: Central Limit Theorem

Given a random variable (RV) with known mean  $\mu$  and known standard deviation  $\sigma$ . We are sampling with size  $n$  and we are interested in two new RVs - the sample mean,  $\bar{X}$ , and the sample sum,  $\Sigma X$ . If the size  $n$  of the sample is sufficiently large, then  $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$  and  $\Sigma X \sim N(n\mu, \sqrt{n}\sigma)$ . If the size  $n$  of the sample is sufficiently large, then the distribution of the sample means and the distribution of the sample sums will approximate a normal distribution regardless of the shape of the population. The mean of the sample means will equal the population mean and the mean of the sample sums will equal  $n$  times the population mean. The standard deviation of the distribution of the sample means,  $\frac{\sigma}{\sqrt{n}}$ , is called the standard error of the mean.

### Definition 2: Standard Deviation

A number that is equal to the square root of the variance and measures how far data values are from their mean. Notation:  $s$  for sample standard deviation and  $\sigma$  for population standard deviation.